Robust Optimal H_∞ Control for Uncertain 2-D Discrete Systems described by the General Model via State feedback Controller

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Abstract

This paper is concerned with the problem of H_∞ control for uncertain two-dimensional (2-D) discrete systems described by the General model (GM). The parameter uncertainty is assumed norm-bounded. A sufficient condition to have an H_∞ noise attenuation for this uncertain 2-D discrete system is given in terms of a certain linear matrix inequality (LMI). A convex optimization problem is proposed to design an optimal H_∞ state feedback controller which ensures stability of the uncertain 2-D discrete system as well as achieving the least value of H_∞ noise attenuation level of resulting closed-loop system. Finally, an illustrative example is given to demonstrate the applicability of proposed approach.

Keywords *Two-dimensional systems; H_∞ control; linear matrix inequality; state feedback controller; general model*

I. INTRODUCTION

Along with the growing modern developing technology in industries, the research on Multivariable systems and Multidimensional signals have received considerable attention due to its practical and theoretical interest in the different fields such as thermal processes in chemical reactor, digital filtering, seismographic data processing, water stream heating, data processing, gas absorption, heat exchanger and pipe furnaces, etc have a natural 2-D representation [1, 24, 25, 26]. Therefore, the analysis and synthesis of 2-D discrete systems is an interesting and challenging task, and it has received much attention, for example different 2-D linear state-space models have been proposed by several researchers such as Attasi, Givone-Roesser, Fornasini-Marchesini (FM) [2, 3, 4], Hinamoto (1997) addresses stability of 2-D systems [27], and Bisiacco (1995) presented the 2-D optimal control theory [28]. To effectively solve the noise and/or disturbance attenuation problem for 2-D systems, Sebek (1993) first addressed the H_∞ control problem for 2-D systems [29] and Du and Xie (2002) established several versions of 2-D bounded real lemma [30].

Stability analysis of system is the main issue of designing any control system. When introducing state-space models of 2-D discrete system, many lyapunov equations are used as powerful tools for stability analysis of 2-D discrete system and errors are inevitable as the actual system parameters would be different than the estimated system parameters. These errors changes in the operating conditions, system aging etc. This may lead to instability and degradation in the performance [33]. Therefore, it is desirable to design a control systems which not only stabilize, but also guarantees an adequate level of performance [16]. One approach to this problem is the so called guaranteed cost control approach. This approach has the advantage of providing an upper bound of a given quadratic cost function and thus the system performance degradation incurred by the uncertainties is guaranteed to be less than this bound. Based on this idea, many significant results have been obtained in the literature [11, 13, 14, 15]. Stability analysis and filter design have been addressed in [5, 6, 7, 8, 9, 12].

In recent years, the problem of H_∞ control is an attractive topic in the theory analysis and practical implementation. In various engineering systems time-delay and uncertainty phenomenon appears several times in various engineering systems such as chemical processing, networked control system, aircraft etc. In a dynamic system, instability, oscillations or performance degraded due to the existences of delays and uncertainties. The system stability and performance are two fundamental requirements in the uncertain control system design. A stable system must have a good dynamic performance such as fast response, effective load rejection, and small overshoot. In the past, much attention has been focused on the study of H_∞ control problems where main goal of designing controller is to stabilize the closed-loop system and the H_∞ norm of the resulting closed-loop transfer function is minimized.

In the system theory, H_∞ norm of the transfer function from external inputs including noise and disturbances to the output is one of the popular performance measurements [17, 18, 19, 20]. Thus, effects of the disturbance on the system performance is reduced. Therefore, time delay and uncertain systems are the most interesting topics in control fields over the decades. A major advantage of H_∞ control is that its performance specification takes account of the worst-case performance for system in terms of system energy gain [18, 20]. This is suitable for system robustness analysis and robust control with modeling uncertainties and disturbances than other specification. In this paper, we analyze system by H_∞ performance measure, which is an upper bound of maximum gain in the space over all frequencies. The advantage of using the H_∞ performance measure is its lesser sensitivity to uncertainty in the exogenous signal. This has motivated the study of robust H_∞ control for uncertain 2-D discrete systems. To the authors' knowledge, the robust H_∞ control for uncertain 2-D discrete systems described by general model (GM) via state feedback controller has not been investigated up to date. GM is one of the best model because it is superset of all state-space models and structurally different from other models.

Therefore, with above motivation, this paper addresses to the problem of H_∞ control for uncertain 2-D discrete Systems described by general model. The approach adopted in this paper is as follows: We have developed a sufficient condition for such a system to have specified H_∞ noise attenuation is first presented via the LMI approach. To design the optimal H_∞ state feedback controller, LMI constraints are formulated such that for closed-loop system, H_∞ noise attenuation level γ is minimized. Finally, the numerical example is given to demonstrate the effectiveness of the proposed method.

Notation

The following notations are used throughout the paper: R^n denotes real vector space of dimension n, $R^{n\times m}$ denotes the set of $n\times m$ real matrices, $\mathbf{0}$ is the null matrix or null vector of appropriate dimension, $\boldsymbol{I}$ is the identity matrix of appropriate dimension, the superscript T stands for matrix transposition, $diag\{...\}$ stands for a block diagonal matrix, $\boldsymbol{G}>\mathbf{0}$ (respectively, $\boldsymbol{G}<\mathbf{0}$) denotes a matrix $\boldsymbol{G}$, which is real symmetric and positive (respectively, negative) definite.

II. PROBLEM FORMULATION AND PRELIMINARIES

This paper deals with the problem of H_∞ control for uncertain 2-D discrete systems described by the GM [3]. Specifically, the system under consideration is given by

$$\begin{aligned}&\boldsymbol{x}(i+1,j+1)\\&=\bar{\boldsymbol{A}}_1\boldsymbol{x}(i,j+1)+\bar{\boldsymbol{A}}_2\boldsymbol{x}(i+1,j)+\bar{\boldsymbol{A}}_0\boldsymbol{x}(i,j)\\&+\bar{\boldsymbol{B}}_1\boldsymbol{w}(i,j+1)+\bar{\boldsymbol{B}}_2\boldsymbol{w}(i+1,j)+\bar{\boldsymbol{B}}_0\boldsymbol{w}(i,j)\\&+\bar{\boldsymbol{C}}_1\boldsymbol{u}(i,j+1)+\bar{\boldsymbol{C}}_2\boldsymbol{u}(i+1,j)+\bar{\boldsymbol{C}}_0\boldsymbol{u}(i,j)\end{aligned}\tag{1.1a}$$

$$\boldsymbol{z}(i,j)=\boldsymbol{H}\boldsymbol{x}(i,j)+\boldsymbol{L}\boldsymbol{w}(i,j)\,.\tag{1.1b}$$

where $0\le i,j\in\mathbf{Z}$ ($\mathbf{Z}$ denotes a set of integer) are horizontal and vertical coordinates, respectively; $\boldsymbol{x}(i,j)\in R^n$ is the state vector, $\boldsymbol{u}(i,j)\in R^m$ is the input vector, $\boldsymbol{z}(i,j)\in R^p$ is the controlled output, $\boldsymbol{w}(i,j)\in R^q$ is the noise input which belongs to $\ell_2\{[0,\infty),[0,\infty)\}$ and

$$\begin{aligned}&\bar{\boldsymbol{A}}_1=(\boldsymbol{A}_1+\Delta\boldsymbol{A}_1),\quad \bar{\boldsymbol{A}}_2=(\boldsymbol{A}_2+\Delta\boldsymbol{A}_2),\quad \bar{\boldsymbol{A}}_0=(\boldsymbol{A}_0+\Delta\boldsymbol{A}_0),\\&\bar{\boldsymbol{B}}_1=(\boldsymbol{B}_1+\Delta\boldsymbol{B}_1),\quad \bar{\boldsymbol{B}}_2=(\boldsymbol{B}_2+\Delta\boldsymbol{B}_2),\quad \bar{\boldsymbol{B}}_0=(\boldsymbol{B}_0+\Delta\boldsymbol{B}_0),\\&\bar{\boldsymbol{C}}_1=(\boldsymbol{C}_1+\Delta\boldsymbol{C}_1),\quad \bar{\boldsymbol{C}}_2=(\boldsymbol{C}_2+\Delta\boldsymbol{C}_2),\quad \bar{\boldsymbol{C}}_0=(\boldsymbol{C}_0+\Delta\boldsymbol{C}_0).\end{aligned}\tag{1.1c}$$

The matrices $\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \boldsymbol{A}_0\in R^{n\times n}$, $\boldsymbol{B}_1,\ \boldsymbol{B}_2,\ \boldsymbol{B}_0\in R^{n\times q}$, $\boldsymbol{C}_1,\ \boldsymbol{C}_2,\ \boldsymbol{C}_0\in R^{n\times m}$, $\boldsymbol{H}\in R^{p\times n}$ and $\boldsymbol{L}\in R^{p\times q}$ are known constant matrices representing the nominal plant. The matrices $\Delta\boldsymbol{A}_1$, $\Delta\boldsymbol{A}_2$, $\Delta\boldsymbol{A}_0$, $\Delta\boldsymbol{B}_1$, $\Delta\boldsymbol{B}_2$, $\Delta\boldsymbol{B}_0$, $\Delta\boldsymbol{C}_1$, $\Delta\boldsymbol{C}_2$, and $\Delta\boldsymbol{C}_0$ represent parameter uncertainties in the system matrices, which are assumed to be of the form

$$\begin{aligned}&\begin{bmatrix}\Delta\boldsymbol{A}_1 & \Delta\boldsymbol{B}_1 & \Delta\boldsymbol{C}_1\end{bmatrix}=\boldsymbol{H}_0\boldsymbol{F}(i,j+1)\begin{bmatrix}\boldsymbol{E}_1 & \boldsymbol{E}_4 & \boldsymbol{E}_7\end{bmatrix},\\&\begin{bmatrix}\Delta\boldsymbol{A}_2 & \Delta\boldsymbol{B}_2 & \Delta\boldsymbol{C}_2\end{bmatrix}=\boldsymbol{H}_0\boldsymbol{F}(i+1,j)\begin{bmatrix}\boldsymbol{E}_2 & \boldsymbol{E}_5 & \boldsymbol{E}_8\end{bmatrix},\\&\begin{bmatrix}\Delta\boldsymbol{A}_0 & \Delta\boldsymbol{B}_0 & \Delta\boldsymbol{C}_0\end{bmatrix}=\boldsymbol{H}_0\boldsymbol{F}(i,j)\begin{bmatrix}\boldsymbol{E}_3 & \boldsymbol{E}_6 & \boldsymbol{E}_9\end{bmatrix}.\end{aligned}\tag{1.1d}$$

Where $\boldsymbol{H}\in R^{p\times n}$, and $\boldsymbol{L}\in R^{p\times q}$ are known constant matrices representing the nominal plant and $\boldsymbol{H}_0\in R^{n\times k}$, $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3\in R^{l\times n}$, $\boldsymbol{E}_4$, $\boldsymbol{E}_5$, $\boldsymbol{E}_6$ $\in R^{l\times q}$ and $\boldsymbol{E}_7$, $\boldsymbol{E}_8$, $\boldsymbol{E}_9\in R^{l\times m}$ are known structural matrices of uncertainty.

$\boldsymbol{F}(i,j+1),\boldsymbol{F}(i+1,j),\boldsymbol{F}(i,j)\in R^{k\times l}$ is an unknown matrix representing parameter uncertainty and satisfies

$$\begin{cases}\boldsymbol{F}^{T}(i,j+1)\boldsymbol{F}(i,j+1)\le \boldsymbol{I},\\ \boldsymbol{F}^{T}(i+1,j)\boldsymbol{F}(i+1,j)\le \boldsymbol{I},\\ \boldsymbol{F}^{T}(i,j)\boldsymbol{F}(i,j)\le \boldsymbol{I}.\end{cases} \quad (1.2)$$

For the 2-D system (1.1), assume a finite set of initial conditions, i.e., there exist positive integers r_1 and r_2 such that

$$\begin{cases}\boldsymbol{x}(i,0)=\boldsymbol{0},\ i>r_1,\\ \boldsymbol{x}(0,j)=\boldsymbol{0},\ j>r_2.\end{cases} \quad (1.3)$$

The following lemmas are essential for our main results.

Lemma 1.1 [21, 22, 23] Let $\boldsymbol{A}\in R^{n\times n}$, $\boldsymbol{H}_0\in R^{n\times k}$, $\boldsymbol{E}\in R^{l\times n}$ and $\boldsymbol{Q}=\boldsymbol{Q}^T\in R^{n\times n}$ be given matrices. Then there exist a positive definite matrix $\boldsymbol{P}$ such that

$$[\boldsymbol{A}+\boldsymbol{H}_0\boldsymbol{F}\boldsymbol{E}]^T\boldsymbol{P}[\boldsymbol{A}+\boldsymbol{H}_0\boldsymbol{F}\boldsymbol{E}]-\boldsymbol{Q}<\boldsymbol{0}$$

for all $\boldsymbol{F}(i,j)$ satisfying $\boldsymbol{F}^T(i,j)\boldsymbol{F}(i,j)\le\boldsymbol{I}$, if and only if there exists a scalar $\varepsilon>0$ such that

$$\begin{bmatrix}-\boldsymbol{P}^{-1}+\varepsilon\boldsymbol{H}_0\boldsymbol{H}_0^T & \boldsymbol{A}\\ \boldsymbol{A}^T & \varepsilon^{-1}\boldsymbol{E}^T\boldsymbol{E}-\boldsymbol{Q}\end{bmatrix}<\boldsymbol{0}. \quad (1.5)$$

Lemma 1.2 [31, 32]

For real matrices $\boldsymbol{M},\ \boldsymbol{L},\ \boldsymbol{Q}$ of appropriate dimension, where $\boldsymbol{M}=\boldsymbol{M}^T$ and $\boldsymbol{Q}=\boldsymbol{Q}^T>\boldsymbol{0}$ then $\boldsymbol{M}+\boldsymbol{L}^T\boldsymbol{Q}\boldsymbol{L}<\boldsymbol{0}$, if and only if

$$\begin{bmatrix}\boldsymbol{M} & \boldsymbol{L}^T\\ \boldsymbol{L} & -\boldsymbol{Q}^{-1}\end{bmatrix}<\boldsymbol{0} \quad (1.6)$$

or equivalently

$$\begin{bmatrix}-\boldsymbol{Q}^{-1} & \boldsymbol{L}\\ \boldsymbol{L}^T & \boldsymbol{M}\end{bmatrix}<\boldsymbol{0}. \quad (1.7)$$

Definition 1: The system (1.1) is asymptotically stable if $\lim_{r\to\infty}X_r=0$ with $\boldsymbol{w}(i,j)=\boldsymbol{0}$, $\boldsymbol{u}(i,j)=\boldsymbol{0}$, and the initial condition (1.3).

Where $X_r=\sup\{\|x(i,j)\|: i+j=r, i,j\in \boldsymbol{Z}\}$.

Definition 2: Consider system (1.1) with $\boldsymbol{u}(i,j)=\boldsymbol{0}$ and the initial condition (1.3). Given a scalar $\gamma>0$, and symmetric positive definite weighting matrices $\boldsymbol{Q}_1,\boldsymbol{Q}_2,\boldsymbol{Q}_3\in R^{n\times n}$, 2-D system (1.1) is said to have an H_∞ noise attenuation γ if it is asymptotically stable and satisfies

$$\boldsymbol{J}=\sup_{0\ne w\in\ell_2}\frac{\|\bar{\boldsymbol{z}}\|_2^2}{\|\bar{\boldsymbol{w}}\|_2^2+D_1(0,j)+D_2(i,0)+D_3(i,j)}<\gamma^2. \quad (1.8)$$

Where

$$\|\bar{\boldsymbol{z}}\|_2^2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\\ \boldsymbol{z}(i,j)\end{bmatrix}\right\|^2,\quad \|\bar{\boldsymbol{w}}\|_2^2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}\right\|^2,$$

$$D_1(0,j)=\sum_{i=0}^{\infty}\boldsymbol{x}^T(0,j+1)\boldsymbol{Q}_1\boldsymbol{x}(0,j+1),$$

$$D_2(i,0)=\sum_{j=0}^{\infty}\boldsymbol{x}^T(i+1,0)\boldsymbol{Q}_2\boldsymbol{x}(i+1,0),\ \text{and}$$

$$D_3(i,j)=\sum_{j=0}^{\infty}\boldsymbol{x}^T(0,j)\boldsymbol{Q}_3\boldsymbol{x}(0,j)+\sum_{i=0}^{\infty}\boldsymbol{x}^T(i,0)\boldsymbol{Q}_3\boldsymbol{x}(i,0).$$

The following theorem presents a sufficient condition for 2-D system (1.1) to ensure the asymptotically stability as well as a specified H_∞ noise attenuation γ.

Theorem 1 Given a positive scalar γ, system (1.1) with $\boldsymbol{u}(i,j)=\boldsymbol{0}$ and initial condition (1.3) has a H_∞ noise attenuation γ if there exist symmetric positive definite matrices $\boldsymbol{P},\boldsymbol{P}_1,\boldsymbol{P}_2\in R^{n\times n}$ satisfying $\boldsymbol{P}_1>\gamma^2\boldsymbol{Q}_1$, $\boldsymbol{P}_2>\gamma^2\boldsymbol{Q}_2$, and $\boldsymbol{0}<\boldsymbol{P}-\boldsymbol{P}_1-\boldsymbol{P}_2<\gamma^2\boldsymbol{Q}_3$, such that the following LMI is feasible:

$$\left(\begin{bmatrix}\bar{A}_1^T\\ \bar{A}_2^T\\ \bar{A}_0^T\\ \bar{B}_1^T\\ \bar{B}_2^T\\ \bar{B}_0^T\end{bmatrix} P \begin{bmatrix}\bar{A}_1^T\\ \bar{A}_2^T\\ \bar{A}_0^T\\ \bar{B}_1^T\\ \bar{B}_2^T\\ \bar{B}_0^T\end{bmatrix}^T + \begin{bmatrix} -P_1+H^TH & 0 & 0 & H^TL & 0 & 0\\ 0 & -P_2+H^TH & 0 & 0 & H^TL & 0\\ 0 & 0 & -P+P_1+P_2+H^TH & 0 & 0 & H^TL\\ L^TH & 0 & 0 & L^TL-\gamma^2 I & 0 & 0\\ 0 & L^TH & 0 & 0 & L^TL-\gamma^2 I & 0\\ 0 & 0 & L^TH & 0 & 0 & L^TL-\gamma^2 I\end{bmatrix}\right) < 0. \quad (1.9)$$

Proof Suppose that there exist $P_1 > 0$, $P_2 > 0$, and $P - P_1 - P_2 > 0$ such that LMI (1.5) holds. We define a Lyapunov function

$$V(x(i,j)) = V_1(x(i,j)) + V_2(x(i,j)) + V_3(x(i,j)). \quad (1.10)$$

Where

$$V_1(x(i,j)) = x^T(i,j)P_1x(i,j)$$

$$V_2(x(i,j)) = x^T(i,j)P_2x(i,j)$$

$$V_3(x(i,j)) = x^T(i,j)(P - P_1 - P_2)x(i,j).$$

Thus, it is confirm that $V(x(i,j))$ is positive.

The increment $\Delta V(i+1, j+1)$ along any trajectory of system (1.1) with $u(i,j) = 0$ and $w(i,j) = 0$ satisfies

$$\Delta V(i+1, j+1)$$
$$= V_1(x(i+1, j+1)) + V_2(x(i+1, j+1)) + V_3(x(i+1, j+1)) - V_1(x(i, j+1)) - V_2(x(i+1, j)) - V_3(x(i,j))$$
$$= \begin{bmatrix} x(i,j+1)\\ x(i+1,j)\\ x(i,j)\end{bmatrix}^T \left(\begin{bmatrix}\bar{A}_1^T\\ \bar{A}_2^T\\ \bar{A}_0^T\end{bmatrix} P \begin{bmatrix}\bar{A}_1^T\\ \bar{A}_2^T\\ \bar{A}_0^T\end{bmatrix}^T + \begin{bmatrix} -P_1 & 0 & 0\\ 0 & -P_2 & 0\\ 0 & 0 & -P+P_1+P_2\end{bmatrix}\right)\begin{bmatrix} x(i,j+1)\\ x(i+1,j)\\ x(i,j)\end{bmatrix}.$$

It follows from the LMI (1.5) that

$$\Delta V(i+1, j+1) \le 0, \text{ i.e.}$$

$$V_1(x(i+1, j+1)) + V_2(x(i+1, j+1)) + V_3(x(i+1, j+1)) \le V_1(x(i, j+1)) + V_2(x(i+1, j)) + V_3(x(i,j)). \quad (1.11)$$

Where equality sign holds only in above (1.11) when $x(i, j+1) = 0$, $x(i+1, j) = 0$, and $x(i,j) = 0$,

Let us assume that

$$D(r) = \{(i,j) : i + j = r : i \ge 0, j \ge 0\} \quad (1.12)$$

For integer $r \ge max\{r_1, r_2\}$, it follows from (1.10) and the initial condition (1.3) that

$$\sum_{i+j\in D(r)} V(x(i,j))$$
$$= \sum_{i+j\in D(r)} V_1(x(i,j)) + V_2(x(i,j)) + V_3(x(i,j))$$

$$= V_1(x(r,0)) + V_1(x(r-1,1)) + V_1(x(r-2,2)) + \ldots + V_1(x(1,r-1)) + V_1(x(0,r))$$
$$+ V_2(x(r,0)) + V_2(x(r-1,1)) + V_2(x(r-2,2)) + \ldots + V_2(x(1,r-1)) + V_2(x(0,r))$$
$$+ V_3(x(r,0)) + V_3(x(r-1,1)) + V_3(x(r-2,2)) + \ldots + V_3(x(1,r-1)) + V_3(x(0,r))$$
$$\ge V_1(x(r+1,0)) + V_1(x(r,1)) + V_1(x(r-1,2)) + \ldots + V_1(x(1,r)) + V_1(x(0,r+1)) + V_2(x(r+1,0))$$
$$+ V_2(x(r,1)) + V_2(x(r-1,2)) + \ldots + V_2(x(1,r)) + V_2(x(0,r+1)) + V_3(x(r+1,0))$$
$$+ V_3(x(r,1)) + V_3(x(r-1,2)) + \ldots + V_3(x(1,r)) + V_3(x(0,r+1)) - V_1(x(-1,r+1))$$
$$- V_2(x(r+1,-1)) - V_3(x(-1,-1)) =$$
$$\sum_{i+j\in D(r+1)} V(x(i,j)) \ge V_1(x(r+2,0)) + V_1(x(r+1,1)) + V_1(x(r,2)) + \ldots + V_1(x(1,r+1)) + V_1(x(0,r+2))$$
$$+ V_2(x(r+2,0)) + V_2(x(r+1,1)) + V_2(x(r,2)) + \ldots + V_2(x(1,r+1)) + V_2(x(0,r+2))$$
$$+ V_3(x(r+2,0)) + V_3(x(r+1,1)) + V_3(x(r,2)) + \ldots +$$

$$V_3(x(1,r+1))+V_3(x(0,r+2))-V_1(x(-1,r+1)) -V_2(x(r+1,-1))-V_3(x(-1,-1))- V_1(x(-2,r+2))-V_2(x(r+2,-2)) -V_3(x(-2,-2))=\sum_{(i+j)\in D(r+2)} V(x(i,j)) \quad (1.13)$$

This implies that the whole energies stored at the points is strictly less than those at the points $\{(i,j): i+j=r+1\}$ and the whole energies stored at the points $\{(i,j): i+j=r+1\}$ is strictly less than those at the points $\{(i,j): i+j=r\}$.

i.e.

$$\boldsymbol{E}\{(i,j): i+j=r+2\} < \boldsymbol{E}\{(i,j): i+j=r+1\} < \boldsymbol{E}\{(i,j): i+j=r\}$$

unless all $x(i,j)=0$. Thus we get

$$\lim_{r\to\infty} \sum_{(i,j)\in D(r)} V\left(x(i,j)\right)=0 \quad (1.14)$$

It follows that

$$\lim_{i+j\to\infty} V\left(\boldsymbol{x}(i,j)\right)=0, \qquad \lim_{i+j\to\infty} \boldsymbol{x}(i,j)=0.$$

Here, we bring to a close that from above the system (1.1) is asymptotically stable.

Now for the $\boldsymbol{H}_\infty$ performance of system (1.1) with control input $\boldsymbol{u}(i,j)=\boldsymbol{0}$ for $\boldsymbol{w}(i,j)\in \ell_2\{[0,\infty),[0,\infty)\}$, we consider

$$\Delta V(i+1,j+1)+\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\\ \boldsymbol{z}(i,j)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\\ \boldsymbol{z}(i,j)\end{bmatrix} -\gamma^2\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}$$

$$=\begin{bmatrix}\boldsymbol{x}(i,j+1)\\ \boldsymbol{x}(i+1,j)\\ \boldsymbol{x}(i,j)\\ \boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}^T\left(\begin{bmatrix}\bar{\boldsymbol{A}}_1^T\\ \bar{\boldsymbol{A}}_2^T\\ \bar{\boldsymbol{A}}_0^T\\ \bar{\boldsymbol{B}}_1^T\\ \bar{\boldsymbol{B}}_2^T\\ \bar{\boldsymbol{B}}_0^T\end{bmatrix}\boldsymbol{P}\begin{bmatrix}\bar{\boldsymbol{A}}_1^T\\ \bar{\boldsymbol{A}}_2^T\\ \bar{\boldsymbol{A}}_0^T\\ \bar{\boldsymbol{B}}_1^T\\ \bar{\boldsymbol{B}}_2^T\\ \bar{\boldsymbol{B}}_0^T\end{bmatrix}^T + \begin{bmatrix}-\boldsymbol{P}_1+\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & -\boldsymbol{P}_2+\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P}+\boldsymbol{P}_1+\boldsymbol{P}_2+\boldsymbol{H}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H}^T\boldsymbol{L}\\ \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{L}^T\boldsymbol{L}-\gamma^2\boldsymbol{I}\end{bmatrix}\right)\begin{bmatrix}\boldsymbol{x}(i,j+1)\\ \boldsymbol{x}(i+1,j)\\ \boldsymbol{x}(i,j)\\ \boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}. \quad (1.15)$$

It follows from LMI (1.5) that

$$\Delta V(i+1,j+1)+\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\\ \boldsymbol{z}(i,j)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{z}(i,j+1)\\ \boldsymbol{z}(i+1,j)\\ \boldsymbol{z}(i,j)\end{bmatrix} -\gamma^2\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}^T\begin{bmatrix}\boldsymbol{w}(i,j+1)\\ \boldsymbol{w}(i+1,j)\\ \boldsymbol{w}(i,j)\end{bmatrix}<0.$$

Therefore, for any integers $T_1, T_2 \to \infty$, we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\Delta V(i+1,j+1)+\begin{bmatrix} z(i,j+1)\\ z(i+1,j)\\ z(i,j)\end{bmatrix}^T\begin{bmatrix} z(i,j+1)\\ z(i+1,j)\\ z(i,j)\end{bmatrix}\right.$$

$$\left.-\gamma^2\begin{bmatrix} w(i,j+1)\\ w(i+1,j)\\ w(i,j)\end{bmatrix}^T\begin{bmatrix} w(i,j+1)\\ w(i+1,j)\\ w(i,j)\end{bmatrix}\right)<0. \qquad (1.16)$$

So, from equation (1.16)

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V(i+1,j+1)+\|\bar{z}\|_2^2-\gamma^2\|\bar{w}\|_2^2<0 \qquad (1.17)$$

$$\|\bar{z}\|_2^2-\gamma^2\|\bar{w}\|_2^2$$
$$<-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V(i+1,j+1)$$
$$=\sum_{j=0}^{\infty}\left[\left(x^T(0,j+1)P_1x(0,j+1)\right)\right]$$
$$+\sum_{i=0}^{\infty}\left[\left(x^T(i+1,0)P_2x(i+1,0)\right)\right]$$
$$+\sum_{i=0}^{\infty}\left[\left(x^T(i,0)(P-P_1-P_2)x(i,0)\right)\right]$$
$$+\sum_{j=0}^{\infty}\left[\left(x^T(0,j)(P-P_1-P_2)x(0,j)\right)\right]. \qquad (1.18)$$

Since $P_1<\gamma^2Q_1$, $P_2<\gamma^2Q_2$, and $P-P_1-P_2<\gamma^2Q_3$, the inequality (1.18) leads to

$$\|\bar{z}\|_2^2<$$
$$\gamma^2\left(\|\bar{w}\|_2^2+\sum_{j=0}^{\infty}\left[x^T(0,j+1)Q_1x(0,j+1)\right]\right.$$
$$+\sum_{i=0}^{\infty}\left[x^T(i+1,0)Q_2x(i+1,0)\right]$$
$$+\sum_{i=0}^{\infty}\left[\left(x^T(i,0)Q_3x(i,0)\right)\right]$$
$$\left.+\sum_{j=0}^{\infty}\left[\left(x^T(0,j)Q_3x(0,j)\right)\right]\right). \qquad (1.19)$$

Therefore, it follows definition (2) that result of theorem (1) is true.

When we consider the case of zero initial condition then the condition for $P_1<\gamma^2Q_1$, $P_2<\gamma^2Q_2$, and $P-P_1-P_2<\gamma^2Q_3$, in theorem 1 are no longer needed it follows from (1.15) that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix} z(i,j+1)\\ z(i+1,j)\\ z(i,j)\end{bmatrix}\right\|^2<\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix} w(i,j+1)\\ w(i+1,j)\\ w(i,j)\end{bmatrix}\right\|^2 \qquad (1.20)$$

i.e., $3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|z(i,j)\|^2-\sum_{i=0}^{\infty}\|z(i,0)\|^2-\sum_{j=0}^{\infty}\|z(0,j)\|^2<$

$$3\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|w(i,j)\|^2-\gamma^2\sum_{i=0}^{\infty}\|w(i,0)\|^2$$

$$-\gamma^2\sum_{j=0}^{\infty}\|w(0,j)\|^2. \qquad (1.21)$$

By considering the zero initial conditions $x(i,0)=x(0,j)=0$, $i,j=0,1,\cdots$. Then, from system (1.1) we have that $z(i,0)=Lw(i,0)$ and $z(0,j)=Lw(0,j)$. Thus, we get from (1.21) that

$$3\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|z(i,j)\|^2<3\gamma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|w(i,j)\|^2$$
$$-\sum_{i=0}^{\infty}w^T(i,0)(\gamma^2I-L^TL)w(i,0)$$
$$-\sum_{j=0}^{\infty}w^T(0,j)(\gamma^2I-L^TL)w(0,j). \qquad (1.18)$$

It may known from (1.5), that $\gamma^2I-L^TL>0$. Thus, for all nonzero $w(i,j)$, we obtain

$$\|z\|_2<\gamma\|w\|_2, \qquad (1.22)$$

where $\|z\|_2=\sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|z(i,j)\|^2}$. and

$$\|w\|_2=\sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|w(i,j)\|^2},$$

It follows from that 2-D Perseval's Theorem [25] that above equation (1.22) is equivalent to

$$\|G(z_1,z_2)\|_\infty=\sup_{\omega_1,\omega_2\in[0,2\pi]}\sigma_{\max}\left[G\left(e^{j\omega_1},e^{j\omega_2}\right)\right]<\gamma, \qquad (1.23)$$

where $\sigma_{\max}(.)$ denotes the maximum singular value of the corresponding matrix and the transfer function from the noise input $w(i,j)$ to the controlled output $z(i,j)$ for the 2-D system (1.1) is

$$\boldsymbol{G}(z_1, z_2) = \boldsymbol{H}(z_1 z_2 \boldsymbol{I}_n - z_2 \bar{\boldsymbol{A}}_1 - z_1 \bar{\boldsymbol{A}}_2 - \bar{\boldsymbol{A}}_0)^{-1}$$
$$\times (z_2 \bar{\boldsymbol{B}}_1 + z_1 \bar{\boldsymbol{B}}_2 + \bar{\boldsymbol{B}}_0) + \boldsymbol{L}. \tag{1.24}$$

Remark 2.1: In this system, controller $\boldsymbol{K}$ and disturbance are playing zero-sum game, in which the cost is $(\|\boldsymbol{z}\|_2^2 - \gamma^2 \|\boldsymbol{w}\|_2^2)$. Our aim is to minimize γ^2 such that the controller can win the game (i.e. achieve negative cost) means

$$(\|\boldsymbol{z}\|_2^2 - \gamma^2 \|\boldsymbol{w}\|_2^2) < 0 \Rightarrow \sup \frac{\|\boldsymbol{z}\|_2^2}{\|\boldsymbol{w}\|_2^2} < \gamma^2.$$

III. H_∞ CONTROLLER DESIGN

Consider system (1.1) and the following controller

$$\boldsymbol{u}(i, j) = \boldsymbol{K}\boldsymbol{x}(i, j) \tag{2.1}$$

The resulting closed-loop system is given by

$$\boldsymbol{x}(i+1, j+1) = (\bar{\boldsymbol{A}}_1 + \bar{\boldsymbol{C}}_1 \boldsymbol{K})\boldsymbol{x}(i, j+1)$$
$$+ (\bar{\boldsymbol{A}}_2 + \bar{\boldsymbol{C}}_2 \boldsymbol{K})\boldsymbol{x}(i+1, j) + (\bar{\boldsymbol{A}}_0 + \bar{\boldsymbol{C}}_0 \boldsymbol{K})\boldsymbol{x}(i, j)$$
$$+ \bar{\boldsymbol{B}}_1 \boldsymbol{w}(i, j+1) + \bar{\boldsymbol{B}}_2 \boldsymbol{w}(i+1, j) + \bar{\boldsymbol{B}}_0 \boldsymbol{w}(i, j)$$
$$\boldsymbol{z}(i, j) = \boldsymbol{H}\boldsymbol{x}(i, j) + \boldsymbol{L}\boldsymbol{w}(i, j). \tag{2.2}$$

Our aim to design a controller (2.1) so that system (2.2) is asymptotically stable, and the H_∞ norm of transfer function (1.24) from the noise input $\boldsymbol{w}(i, j)$ to the controlled output $\boldsymbol{z}(i, j)$ for the closed-loop system (2.2) has a specified H_∞ noise attenuation γ, and the controller (2.1) is said to be a γ -suboptimal H_∞ controller for system (1.1).

Theorem 2 Given scalars $\gamma > 0$, $\varepsilon > 0$ and if there exist a matrices $\boldsymbol{N} \in R^{m \times n}$ and symmetric positive definite matrices $\bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}_1, \bar{\boldsymbol{P}}_2 \in R^{n \times n}$ such that the following LMI is feasible:

$$\left[\begin{array}{cccccccccccccccc}
-\bar{P}_1 & 0 & 0 & 0 & 0 & 0 & \bar{P}A_1' + N'C_1' & \bar{P}H' & 0 & 0 & \bar{P}E_1' & E_7'N' & 0 & 0 & 0 & 0 \\
0 & -\bar{P}_1 & 0 & 0 & 0 & 0 & \bar{P}A_2' + N'C_2' & 0 & \bar{P}H' & 0 & \bar{P}E_2' & 0 & E_8'N' & 0 & 0 & 0 \\
0 & 0 & -\bar{P} + \bar{P}_1 + \bar{P}_2 & 0 & 0 & 0 & \bar{P}A_0' + N'C_0' & 0 & 0 & \bar{P}H' & \bar{P}E_3' & 0 & 0 & E_9'N' & 0 & 0 \\
0 & 0 & 0 & -\gamma^2 I & 0 & 0 & B_1' & L' & 0 & 0 & E_4' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\gamma^2 I & 0 & B_2' & 0 & L' & 0 & E_5' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\gamma^2 I & B_0' & 0 & 0 & L' & E_6' & 0 & 0 & 0 & 0 & 0 \\
A_1\bar{P} + C_1 N & A_2\bar{P} + C_2 N & A_0\bar{P} + C_0 N & B_1 & B_2 & B_0 & -\bar{P} + \varepsilon H_0 H_0' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
H\bar{P} & 0 & 0 & L & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H\bar{P} & 0 & 0 & L & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H\bar{P} & 0 & 0 & L & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
E_1\bar{P} & E_2\bar{P} & E_3\bar{P} & E_4 & E_5 & E_6 & 0 & 0 & 0 & 0 & -\varepsilon I & 0 & 0 & 0 & 0 & 0 \\
NE_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I & 0 & 0 & 0 & 0 \\
0 & NE_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I & 0 & 0 & 0 \\
0 & 0 & NE_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I
\end{array}\right] < 0. \tag{2.3}$$

(2.2)

Then the closed-loop system (2.1) has a specified H_∞ noise attenuation γ and

$$\boldsymbol{u}(i, j) = \boldsymbol{N}\bar{\boldsymbol{P}}^{-1}\boldsymbol{x}(i, j) \tag{2.4}$$

is γ -suboptimal H_∞ state feedback controller for 2-D system (1.1).

Proof By applying theorem 1 and Lemma (1.1, 1.2), a sufficient condition for the closed-loop system (2.2) to have a specified H_∞ noise attenuation γ is that exist $\boldsymbol{P}_1 > \boldsymbol{0}$, $\boldsymbol{P}_2 > \boldsymbol{0}$, and $\boldsymbol{P} - \boldsymbol{P}_1 - \boldsymbol{P}_2 > \boldsymbol{0}$ such that

$$\left[\begin{array}{cccccc}
-P^{-1} + \varepsilon H_0 H_0^T & 0 & 0 & 0 & A_1 + C_1 K & A_2 + C_2 K \\
0 & -I & 0 & 0 & H & 0 \\
0 & 0 & -I & 0 & 0 & H \\
0 & 0 & 0 & -I & 0 & 0 \\
A_1^T + K^T C_1^T & H^T & 0 & 0 & -P_1 + \varepsilon^{-1} E_1^T E_1 + \varepsilon^{-1} K^T E_7^T E_7 K & \varepsilon^{-1} E_1^T E_2 \\
A_2^T + K^T C_2^T & 0 & H^T & 0 & \varepsilon^{-1} E_2^T E_1 & -P_2 + \varepsilon^{-1} E_2^T E_2 + \varepsilon^{-1} K^T E_8^T E_8 K \\
A_0' + K^T C_0^T & 0 & 0 & H^T & \varepsilon^{-1} E_3^T E_1 & \varepsilon^{-1} E_3^T E_2 \\
B_1^T & L^T & 0 & 0 & \varepsilon^{-1} E_4^T E_1 & \varepsilon^{-1} E_4^T E_2 \\
B_2^T & 0 & L^T & 0 & \varepsilon^{-1} E_5^T E_1 & \varepsilon^{-1} E_5^T E_2 \\
B_0^T & 0 & 0 & L^T & \varepsilon^{-1} E_6^T E_1 & \varepsilon^{-1} E_6^T E_2
\end{array}\right.$$

$$\left.\begin{matrix} \boldsymbol{A}_0+\boldsymbol{C}_0\boldsymbol{K} & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \boldsymbol{B}_0 \\ \mathbf{0} & \boldsymbol{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{L} & \mathbf{0} \\ \boldsymbol{H} & \mathbf{0} & \mathbf{0} & \boldsymbol{L} \\ \varepsilon^{-1}\boldsymbol{E}_1^T\boldsymbol{E}_3 & \varepsilon^{-1}\boldsymbol{E}_1^T\boldsymbol{E}_4 & \varepsilon^{-1}\boldsymbol{E}_1^T\boldsymbol{E}_5 & \varepsilon^{-1}\boldsymbol{E}_1^T\boldsymbol{E}_6 \\ \varepsilon^{-1}\boldsymbol{E}_2^T\boldsymbol{E}_3 & \varepsilon^{-1}\boldsymbol{E}_2^T\boldsymbol{E}_4 & \varepsilon^{-1}\boldsymbol{E}_2^T\boldsymbol{E}_5 & \varepsilon^{-1}\boldsymbol{E}_2^T\boldsymbol{E}_6 \\ -\boldsymbol{P}+\boldsymbol{P}_1+\boldsymbol{P}_2+\varepsilon^{-1}\boldsymbol{E}_3^T\boldsymbol{E}_3+\varepsilon^{-1}\boldsymbol{K}'\boldsymbol{E}_9'\boldsymbol{E}_9\boldsymbol{K} & \varepsilon^{-1}\boldsymbol{E}_3^T\boldsymbol{E}_4 & \varepsilon^{-1}\boldsymbol{E}_3^T\boldsymbol{E}_5 & \varepsilon^{-1}\boldsymbol{E}_3^T\boldsymbol{E}_6 \\ \varepsilon^{-1}\boldsymbol{E}_4^T\boldsymbol{E}_3 & -\gamma^2\boldsymbol{I}+\varepsilon^{-1}\boldsymbol{E}_4^T\boldsymbol{E}_4 & \varepsilon^{-1}\boldsymbol{E}_4^T\boldsymbol{E}_5 & \varepsilon^{-1}\boldsymbol{E}_4^T\boldsymbol{E}_6 \\ \varepsilon^{-1}\boldsymbol{E}_5^T\boldsymbol{E}_3 & \varepsilon^{-1}\boldsymbol{E}_5^T\boldsymbol{E}_4 & -\gamma^2\boldsymbol{I}+\varepsilon^{-1}\boldsymbol{E}_5^T\boldsymbol{E}_5 & \varepsilon^{-1}\boldsymbol{E}_5^T\boldsymbol{E}_6 \\ \varepsilon^{-1}\boldsymbol{E}_6^T\boldsymbol{E}_3 & \varepsilon^{-1}\boldsymbol{E}_6^T\boldsymbol{E}_4 & \varepsilon^{-1}\boldsymbol{E}_6^T\boldsymbol{E}_5 & -\gamma^2\boldsymbol{I}+\varepsilon^{-1}\boldsymbol{E}_6^T\boldsymbol{E}_6 \end{matrix}\right]$$

$$< \mathbf{0}. \qquad (2.5)$$

Pre-multiplying and post-multiplying both sides of the matrix inequality (2.5) by $diag\left\{\boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{P}^{-1}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}\right\}$ and applying schur Compliments and denoting $\bar{\boldsymbol{P}} = \boldsymbol{P}^{-1}$, $\bar{\boldsymbol{P}}_1 = \bar{\boldsymbol{P}}\boldsymbol{P}_1\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{P}}_2 = \bar{\boldsymbol{P}}\boldsymbol{P}_2\bar{\boldsymbol{P}}$, and $\boldsymbol{N} = \boldsymbol{K}\bar{\boldsymbol{P}}$, it follows that the inequality (2.5) is equal to the LMI (2.3).This completes the proof.

Remark 2.2: It should be observed matrix inequality (2.3) is linear for $\varepsilon > 0$ and $\boldsymbol{N} \in R^{m\times n}$, positive definite matrices $\bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}_1, \bar{\boldsymbol{P}}_2 \in R^{n\times n}$ so we can solve it using Matlab LMI toolbox [34]. If solution is feasible then state feedback controller can be obtain as $\boldsymbol{u}(i,j) = \boldsymbol{K}\boldsymbol{x}(i,j) = \boldsymbol{N}\bar{\boldsymbol{P}}^{-1}\boldsymbol{x}(i,j),$ i.e. Robust control is realized via state feedback.

Furthermore, by solving the following optimization problem:

$$\textbf{minimize}\ \gamma^2 \qquad (2.6)$$

subject to (2.3),

we can obtain $\boldsymbol{H}_\infty$ controller which ensures that the $\boldsymbol{H}_\infty$ noise attenuation γ of the resulting closed-loop system is minimized. This controller (2.1) is called as the optimal $\boldsymbol{H}_\infty$ controller for the system (1.1).

IV. AN ILLUSTRATIVE EXAMPLE

Consider the thermal processes in chemical reactors, heat exchangers and pipe furnaces, which can be expressed in the following partial differential equation [1].

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0 T(x,t) + bu(x,t). \qquad (3.1)$$

Where $T(x,t)$ is the temperature at x (space) $\in\left[0,\ x_f\right]$ and t (time) $\in\left[0,\ \infty\right], u(x,t)$ is the input function, and a_0, b are real coefficients.

Denote $x^T(i,j) = \left[T^T(i-1,j)\quad T^T(i,j)\right],$ it is easy to verify that equation (3.1) can be converted into a 2-D General model (1.1) with

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1-\frac{\Delta t}{\Delta x}-a_0\Delta t \end{bmatrix},$$

$$\boldsymbol{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \boldsymbol{C}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \boldsymbol{C}_2 = \begin{bmatrix} 0 \\ b\Delta t \end{bmatrix},\ \boldsymbol{C}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (3.2)$$

Let $\Delta t = 0.1, \Delta x = 0.4, a_0 = 1,$ and $b = 0.4.$ By considering, the problem of $\boldsymbol{H}_\infty$ disturbance attenuation, and uncertainty, the thermal process is modeled in the form (1.1) with

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 0.004 \end{bmatrix},\quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 0.004 \end{bmatrix},\quad \boldsymbol{B}_0 = \begin{bmatrix} 0 \\ 0.004 \end{bmatrix},$$

$$\boldsymbol{H} = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix},\quad \boldsymbol{H}_0 = \begin{bmatrix} 0.001 & 0.002 \\ 0 & 0 \end{bmatrix},\quad \boldsymbol{L} = 0.5,$$

$$\boldsymbol{E}_1 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix},\ \boldsymbol{E}_2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$\boldsymbol{E}_3 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix},\quad \boldsymbol{E}_4 = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix},$$

$$\boldsymbol{E}_5 = \begin{bmatrix} -0.007 \\ 0 \end{bmatrix},\ \boldsymbol{E}_6 = \begin{bmatrix} -0.007 \\ 0 \end{bmatrix},\quad \boldsymbol{E}_7 = \begin{bmatrix} -0.001 \\ 0 \end{bmatrix},$$

$$\boldsymbol{E}_8 = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix},\ \boldsymbol{E}_9 = \begin{bmatrix} 0.003 \\ 0 \end{bmatrix}. \qquad (3.3)$$

Solving the optimization problem (2.6) using MINCX from LMI toolbox, we obtain optimal $\boldsymbol{H}_\infty$ noise attenuation $\gamma =$ 0.5003 and the optimal $\boldsymbol{H}_\infty$ state feedback controller

$$\boldsymbol{u}(i,j) = \left[-7.4815 \quad -18.9786\right]\boldsymbol{x}(i,j) \qquad (3.4)$$

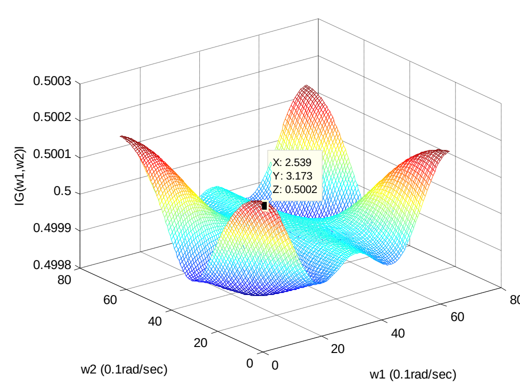

Figure 1. The Frequency Response

Figure 1 shows that the peak value of the frequency response of the closed-loop system is found to be 0.5002, which is below the optimal level of performance 0.5003.

V. CONCLUSION

This paper has presented a solution to the problem of H_∞ Control for 2-D Discrete Uncertain Systems described by the General Model. A sufficient condition for 2-D discrete uncertain system to have a specified H_∞ noise attenuation is proposed in terms of certain LMI. By solving a convex optimization problem the optimal H_∞ controller is obtained. One example is also given to illustrate the applicability of the proposed approach.

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