Results Concerning Acyclic Chromatic Numbers of Graphs

I.H.Nagarajarao[#1], B.L.V.Vinay Kumar[*2]

#1 Director & Sr.Prof.G.V.P College For Degree And P.G.Courses

Rushikonda, Visakhapatnam-530045

[#2] Asst.Professor ,Department of CSE ,G.V.P. College of Engineering For Women

Madhurwada,Visakhapatnam-530041

***Abstract*—** The acyclic chromatic numbers of standard graphs and of their tensor products are evaluated.

***Keywords*—**Chromatic number, k-coloring, Tensor product

I. Introduction

The concept of chromatic number of a graph plays a vital role in colouring problems. The five colour result and the four colour conjecture are very prominent ones in Graph Colouring techniques. Vince[6] introduced a natural generalization of graph colouring and proved some basic results that have enhanced further interest in these problems. Alon et al.[1] introduced the concept of acyclic Colouring. This has useful applications in computational problems. We considered this concept for standard graphs.

Throughout this paper, by a graph we mean a non empty, finite, connected and simple graph.

For standard notions, we refer Bondy and Murthy[2].

II. PRELIMINARIES

2.1. Definition[2] : The chromatic number of a graph G, denoted by $\chi(G)$,is the least positive integer k such that G has a proper $k-(vertix)$ Colouring.
By a proper k-Colouring of a graph G, we mean that the vertices of G are coloured with k colours such that no two adjacent vertices of G receive the same colour in the colouring.

2.2 Definition [1]: A proper k-colouring of a graph G is said to be acyclic iff (if and only if) every sub graph of G induced by two colours (colour classes of the k-colouring) is acyclic (i.e. the sub graph does not contain any cycle) .

2.3. Definition[1] : The acyclic chromatic number of G, denoted by $\chi_a(G)$, is the least positive integer k such that G has an acyclic k-Colouring.

2.4. Convention: For the graph G with single vertex, i.e.trvial graph, we take $\chi_a(G)=1=\chi(G)$.

2.5. Result: If G is any acyclic graph, then $\chi_a(G)=\chi(G)$.
The result follows trivially.

2.6. Examples: (i) For the path $P_n(n\geq 2), \chi_a(P_n)=\chi(P_n)=2$.

(ii) For a tree T with at least two vertices, $\chi_a(T)=\chi(T)=2$.

Both P_n and T are acyclic and $\chi(P_n)=\chi(T)=2$;any subgraph of either P_n or T induced by two colours is itself.

III. ACYCLIC CHROMATIC NUMBER OF STANDARD GRAPHS

3.1. Theorem: For the complete graph $K_n(n\geq 2), \chi_a(K_n)=n=\chi(K_n)$.

Proof :Since K_2 is acyclic follows that $\chi_a(K_2)=2=\chi(K_2)$.

Let $n\geq 3$ by the definition of K_n any two vertices of K_n are adjacent and hence follows that $\chi(K_n)=n$ and the subgraph generated by any two vertices (Colour classes) is K_2 ,which is acyclic. So $\chi_a(K_n)=n=\chi(K_n)$.

3.2. Theorem: For the cycle $C_n(n\geq 3)$ (any cycle has at least three vertices)

$$\chi_a(C_n)=3=\begin{cases}\chi(C_n) \text{ if n is odd,}\\ =\chi(C_n)+1 \text{ if n is even}\end{cases}$$

Proof : Let the vertex set V of C_n be $\{v_1,v_2,\ldots v_n\}$.

Case 1: Suppose n is odd.

If $n=3$ then $C_3=K_3$ and so $\chi_a(C_3)=\chi_a(K_3)=3=\chi(K_3)=\chi(C_3)$.

Let $n\geq 5$. We partition V as follows.

Denote $V_1=\{v_1,v_3,\ldots v_{n-2}\},V_2=\{v_2,v_4,\ldots v_{n-1}\}$ and $V_3=\{v_n\}$.Observe that no two vertices of either V_1 or V_2 are adjacent $(\text{in } C_n)$ and the single vertex v_n is adjacent to both v_1 and v_{n-1} where $v_1\in V_1$ and $v_{n-1}\in V_2$ .So for having a proper colouring we have to assign distinct colours to the V_1 class, V_2 class and V_3 class. Thus a minimum of three colours are required to have proper colouring. So $\chi(C_n)=3$ Further, the subgraph of C_n induced by any two colours are $G[\{V_1\}\cup\{V_2\}],G[\{V_1\}\cup\{V_3\}]$ and $G[\{V_2\}\cup\{V_3\}]$; these are proper subgraph of C_n .Since C_n is the only cycle in C_n it follows that these sub graphs are acyclic and hence $\chi_a(C_n)=3$.

$\Rightarrow\chi_a(C_n)=3=\chi(C_n)$ When n is odd and >1.

Case 2: Suppose n is even.

Now let $V_1=\{v_1,v_3,\ldots v_{n-2}\}$ and $V_2=\{v_2,v_4,\ldots v_n\}$. $\{V_1,V_2\}$ gives rise a partition of V such that no two vertices of either V_1 or V_2 are adjacent in C_n .Hence follows that $\chi(C_n)=2$ .Now any sub graph of C_n induced by two colours is nothing but C_n which is a cycle.

Now take any $v\in V_1\cup V_2$ .Assign a third colour to v and one colour to $V_1-\{v\}$ and second colour to $V_2-\{v\}$ $(V_j-\{v\}=V_j \text{ if } v\notin V_j\,(j=1,2))$.

Denote $V_3=\{v\}$ .Now any subgraph of C_n induced by $(V_1-\{v\})\cup V_3$ or $(V_2-\{v\})\cup V_3$ is a proper subgraph of C_n and hence acyclic. Hence $\chi_a(C_n)=3=2+1=\chi(C_n)+1$.

This completes the proof of the theorem.

3.3. Theorem :If G is any bipartite graph with at least three vertices then

(i) $\chi_a(G)=2=\chi(G)$ if G doesnot contain any even cycle

and (ii) $\chi_a(G)=3=2+1=\chi(G)+1$ If G contains an even cycle.

Proof: Let G be a bipartite graph with a bipartition $\{V_1, V_2\}$ of the vertex set of G.

So no two vertices of either V_1 or V_2 are adjacent in G.We know that G contains no odd cycles $(see\ [2])$.

Case 1: Suppose G has no even cycles.

Hence follows that G is acyclic. Assigning one colour to the vertices of V_1 and another colour to the vertex of V_2,we get a 2-proper colouring of G.Now follows that

$\chi_a(G) = 2$, since G is acyclic; Thus $\chi_a(G) = \chi(G) = 2$.

Case 2: Suppose G has an even cycle.

Let there be m even cycles; say $C_1, \ldots C_m$. Let v_j be a vertex of $C_j\ (j = 1, \ldots m)$

Denote $V_3 = \{v_1, \ldots v_m\}$.

Assign one colour to the vertices of $V_1 - V_3$,a second colour to the vertices of $V_2 - V_3$ and a third colour to the vertices of V_3.

Now all the sub graphs generated by either $\{V_1 - V_3\} \cup \{V_3\}, \{V_2 - V_3\} \cup \{V_3\}$ and $\{V_1 - V_3\} \cup \{V_2 - V_3\}$ are acyclic.(by the selection of V_3) .

So follows that $\chi_a(G) = 3 = \chi(G) + 1$.

This completes the proof of the theorem.

3.4. Theorem: For the star $S_{n+1}(n \geq 1), \chi_a(S_{n+1}) = 2 = \chi(S_{n+1})$.

Proof: Let $V(S_{n+1}) = \{v_0, v_4 \ldots v_n\}$ where v_0 is the centre of S_{n+1}.

This is a bipartite graph with a bipartition $\{V_1, V_2\}$,where $V_1 = \{v_0\}\ and\ V_2 = \{v_{1,} \ldots v_n\}$ .Since the graph has no cycles and $\chi(S_{n+1}) = 2$ it follows that $\chi_a(S_{n+1}) = 2 = \chi(S_{n+1})$.

3.5.Theorem : For the wheel $W_{n+1}(n \geq 3)$

$$\chi_a(W_{n+1}) = 4 = \begin{cases} \chi(W_{n+1}) + 1 \text{ if n is even,} \\ \chi(W_{n+1}) \text{ if n is odd.} \end{cases}$$

Proof : Let the vertex set of W_{n+1} be $\{v_0, v_1, v_2 \ldots v_n\}$, where v_0 is the center of W_{n+1}.

We can write $W_{n+1} = K_1 \vee C_n (\text{see } [2])$

where $V(K_1) = \{v_0\}\ and\ V(C_n) = \{v_1, v_2, v_3 \ldots v_n\}$.

Case 1: Suppose n is even. Since $\chi(C_n)=2$;by assigning a third colour to $\{v_0\}$,

We get a proper 3-colouring to W_{n+1} and further $\chi(W_{n+1})=3$

By assigning a fourth colour to a vertex of C_n ,we get a proper 4-colouring of W_{n+1} such that any sub graph of W_{n+1} induced by two colour classes is acyclic.

Hence $\chi_a(W_{n+1})=4=\chi(W_{n+1})+1$.

Case 2: Suppose n is odd. Since $\chi(C_n)=3$,by assigning a fourth colour to the vertex v_0 ,we get a proper 4-colouring of W_{n+1} and so $\chi(W_{n+1})=4$ .Further ,we observe that any subgraph of W_{n+1} induced by two colour classes is acyclic.(we need at least three colour classes to a cycle).Hence follows that $\chi_a(W_{n+1})=4=\chi(W_{n+1})$,when n is odd.

This completes the proof of the theorem.

IV. Chromatic Numbers of Tensor Products of Standard Graphs

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We first give the following:

4.1. Definition: [5] Let G, H be vertex disjoint graphs. Then the Tensor product of G and H,denoted by $G \wedge H$,is the graph whose vertex set is $\{(\mathrm{u},v):\mathrm{u}\in V(G), v\in V(H)\}$ and the edge set being the set of all elements of the form $(u,v)(u',v')$ where $u,u'\in V(G), v,v'\in V(H)$ and $uu'\in E(G) \,\&\, vv'\in E(H)$.

(Observe that if G and H are simple graph then so is $G \wedge H$).

4.2.Result:[5] If G_1 and G_2 are connected graphs then $G_1 \wedge G_2$ is connected iff either G_1 or G_2 contain an odd cycle.

4.3.Result [3] : For $n\geq 3, K_2 \wedge K_n$ is a connected ,bipartite graph with 2n vertices and $n(n-1)$ edges.Further $K_2 \wedge K_3 = C_6$.

4.4. Theorem: For $n\geq 3, \chi_a(K_2 \wedge K_n)=3=\chi(K_2 \wedge K_n)+1$.

Proof: By Results (4.2)and(4.3) , it follows that $K_2 \wedge K_n$ is connected and bipartite.

Let $V(K_2)=\{u_1,u_2\}$ *and* $V(K_n)=\{v_1,v_2,v_3,...v_n\}$.

A bipartition of $V(K_2 \wedge K_n)$ is $\{V_1,V_2\}$

where $V_1=\{(u_1,v_j): j=1,2,...n\}$ and $V_2=\{(u_2,v_j): j=1,2,...n\}$.

Case (i): Let $n=3$

By theorem [3.2],it follows that

$$\chi_a(K_2 \wedge K_3)=\chi_a(C_6)=\chi(C_6)+1=\chi(K_2 \wedge K_3)+1.$$

Case 2: Let $n \geq 4$;by assigning one colour to the class V_1 and another colour to the class V_2 we get a 2-proper colouring of $K_2 \wedge K_n$ and this is a minimum proper colouring ,so $\chi(K_2 \wedge K_n) = 2$ .Clearly $\{(u_1v_1),(u_2,v_2),(u_1v_3),(u_2v_4),(u_1v_1)\}$ is a cycle in $K_2 \wedge K_n$ .Let there be 'k' cycles say $W_1, W_2, \ldots W_k$ in $K_2 \wedge K_n$ .Let $x_i \in V(W_i)(i = 1,2,\ldots k)$ Denote $V_3 = \{x_i : i = 1,2,\ldots m\}$;assign a third colour to the class V_3 .Now as in the proof of Theorem (3.2),we get that $\chi_a(K_2 \wedge K_n) = 3 = \chi(K_2 \wedge K_n) + 1$.

This completes the proof of the theorem.

4.5. Result:[3]: If m,n are integers such that min $\{m,n\} \geq 2$ and max $\{m,n\} \geq 3$ then $K_m \wedge K_n$ is a simple,connected, $(m-1)(n-1)$-regular graph with mn vertices and hence $\frac{1}{2}mn(m-1)(n-1)$ edges. (A graph is said to be r-regular if and only iff the degree of each vertex is r).

4.6.Theorem: For integers $m,n \geq 2$ and $max\{m,n\} \geq 3$, $\chi(K_m \wedge K_n) = \min\{m,n\}$.

Proof: By the above result $K_m \wedge K_n$ is a connected graph. Let $V(K_m) = \{u_1, u_2 \ldots u_m\}$

and $V(K_n) = \{v_1, v_2 \ldots v_n\}$.

Now $V(K_m \wedge K_n) = \{(u_i, v_j) : i = 1,2,\ldots m; j = 1,2,\ldots n\}$ since K_m, K_n are complete graphs follows that $(u_i, v_j)\, and\, (u_i', v_j')$ are adjacent in $K_m \wedge K_n$ for $i, i' \in \{1,2,\ldots,m\}, j, j' \in \{1,2,\ldots,n\}$ with $i' \neq i$ and $j' \neq j$.

Without loss of generality we can assume that $m = \min\{m,n\}$

Denote $V_i = \{(u_i, v_j : j = 1,2,\ldots n)\}(i = 1,2,\ldots m)$

By the definition of $K_m \wedge K_n$,it follows that $\{V_i : i = 1,\ldots m\}$ are all independent sets (no two elements of V_i are adjacent in $K_m \wedge K_n$).By the selection of 'm',it follows that this is the smallest sub division of $V(K_m \wedge K_n)$;Assignig distint (m)colours to these V_i's ,it follows that this is a proper colouring of $K_m \wedge K_n$ and m is the smallest positive integer of the proper colourings .So by definition ,it follows that $\chi(K_m \wedge K_n) = m = \min\{m,n\} = \min\{\chi(K_m)\chi(K_n)\}$ (by Theorem (3.1)).

4.7. Observation: $\chi_a(K_3 \wedge K_3) = 3 = \chi(K_3 \wedge K_3)$,since the sub graph generated by any two colour classes of $K_3 \wedge K_3$ is acyclic.

4.8. Open Problem: To find the value of $\chi(K_m \wedge K_n)$, when $m,n \geq 3$ and one of m,n is ≥ 4

4.9.Result[4]:For $m,n \geq 3$, $C_m \wedge C_n$ is a simple ,4-regular,connected bipartite graph iff one of m,n is even and other is odd.

4.10.Theorem :For $m,n \geq 3$ and m,n are of different parity (one is even and the other is odd), $\chi_a(C_m \wedge C_n) = 3 = \chi(C_m \wedge C_n) + 1$.

Proof : By the above result ,it follows that $C_m \wedge C_n$ is a connected, bipartite graph with a bipartition,say $\{V_1, V_2\}$ of the vertex set.So,by assigning two distint colours to the classes V_1 and V_2 ,we get a 2-proper colouring of $C_m \wedge C_n$ .Hence $\chi(C_m \wedge C_n) = 2$.

Let $V(C_m) = \{u_1, u_2, ... u_m\}$ and $V(C_n) = \{v_1 v_2, ... v_n\}$ ..

Without loss of generality we can assume that m is odd $\Rightarrow n$ is even

Let $V_1 = \{(u_i, v_j) : i = 1, 2, ... m; j = 1, 3, 5 ... n-1\}$ and

$V_2 = \{(u_i, v_j) : i = 1, 2, ... m; j = 2, 4, ... n\}$.

Now $\{(u_1 v_1), (u_2, v_2)(u_1, v_3)(u_2, v_4), ... (u_1, v_{n-1}), (u_2, v_n)(u_1, v_1)\}$ is a cycle in $C_m \wedge C_n$

Let there be k cycles in $C_m \wedge C_n$ .As in the proof of theorem (4.6),by choosing one vertex from each of the cycles and assigning a third colour to these vertices, we get a proper 3-colouring for the vertices of $C_m \wedge C_n$ .such that any subgraph generated by two colour classes is acyclic. Hence follows that $\chi_a(C_m \wedge C_n) = 3 = \chi_a(C_m \wedge C_n) + 1$.

4.11.Remark: Since $P_m, P_n (m,n \geq 2)$ are connected and acyclic, by result (4.2)it follows that $P_m \wedge P_n$ is disconnected.So ,we are not discussing about acyclic chromatic number of such graphs.

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